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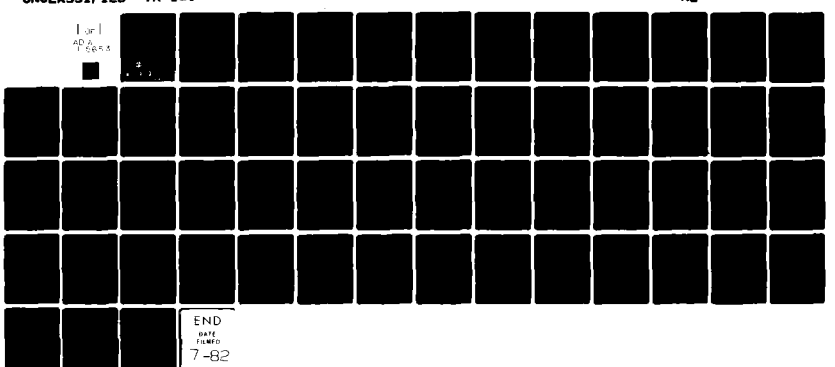

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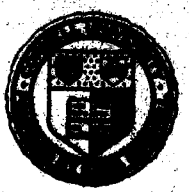

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CLOSED ADAPTIVE SEQUENTIAL PROCEDURES FOR SELECTING THE BEST OF $k \geq 2$ BERNOULLI POPULATIONS

by

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and

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Abstract

The goal of selecting that one of $k \geq 2$ Bernoulli populations which has the largest single-trial "success" probability $p_{[k]} = \max\{p_1,\ldots,p_k\}$ is treated. Consideration is restricted to procedures which take no more than n observations from any one of the k populations. One such procedure is the single-stage procedure of Sobel and Huyett [1957] which takes exactly n observations from each of the k populations. We propose a one-at-a-time adaptive sampling rule (R^*) which when used in conjunction with a particular stopping rule (S^*) and terminal decision rule (T^*) achieves the same probability of a correct selection as does the single-stage procedure uniformly in $\underset{\sim}{p} = (p_1,\ldots p_k)$. Letting N denote the random total number of observations to terminate sampling using the procedure (R^*,S^*,T^*) we show that $n \leq N \leq kn-1$; for $p_{[k]} \to 0$ we have $P\{N = kn-1 \mid \underset{\sim}{p}\} \to 1$ while for $p_{[1]} \to 1$ we have $P\{N = n \mid \underset{\sim}{p}\} \to 1$. For $k = 2$ the sampling rule R^* (the conjugate sampling rule $\bar{R}^*$) which is stationary is optimal in the sense that it minimizes $E\{N|(p_1,p_2)\}$ uniformly in (p_1,p_2) for $p_1 + p_2 > 1$ $(p_1 + p_2 < 1)$ among all sampling rules which use (S^*,T^*) and which take no more than n observations from either population; R^* has additional optimal properties for $k = 2$. The procedure (R^*,S^*,T^*) is generalized for $k > 2$ to accommodate such goals as "Selecting the s $(1 \leq s \leq k-1)$ "best" Bernoulli populations with regard to order," and is shown to have desirable properties for these goals as well. Some conjectures are made concerning the optimality of (R^*,S^*,T^*) for $k > 2$. The performance of (R^*,S^*,T^*) is compared for $k \geq 2$ with that of other sequential selection procedures that have been proposed in the literature. An extensive bibliography is included.

Note

This paper will appear in the Proceedings of the Third Purdue Symposium on Statistical Decision Theory and Related Topics (Eds. S.S. Gupta and J. Berger) held at Purdue University June 1-5, 1981.

1. Introduction

Let Π_i $(1 \leq i \leq k)$ denote $k \geq 2$ Bernoulli populations with corresponding single-trial "success" probabilities p_i. Denote the ordered values of the p_i by $p_{[1]} \leq \cdots \leq p_{[k]}$; the values of the p_i and the pairing of the Π_i with the $p_{[j]}$ $(1 \leq i, j \leq k)$ are assumed to be completely unknown.

Statistical procedures for the problem of selecting the "best" population, i.e., the one associated with $p_{[k]}$, have received considerable attention in recent years. In a fundamental paper, Sobel and Huyett [1957] proposed a single-stage procedure employing the indifference-zone approach of Bechhofer [1954] with the "distance measure" $\Delta_{i,j} = p_i - p_j$; their procedure was shown by Hall [1959] to have the optimum property of being "most economical" among single-stage procedures. Paulson [1967], [1969], using the distance measures $\Delta_{i,j}$ and p_i/p_j, proposed the first sequential procedure for this problem. His open procedure permitted the elimination of "non-contending" populations; it employed a fixed number of stages with a random number of observations per stage, the total number of observations (N) required for termination being an unbounded random variable. Bechhofer, Kiefer and Sobel [1968], Section 12.6.1.4, using the distance measure $p_i(1-p_j)/p_j(1-p_i)$ (and $\Delta_{i,j}$) also proposed an open sequential procedure employing a vector-at-a-time (VT) sampling rule.

Spurred on by the potential of application of such methods in clinical trials and related areas, there followed a period of considerable research activity focusing on sequential procedures for this problem; these studies were spearheaded initially by Milton Sobel, George Weiss, David Hoel and their collaborators: Sobel and Weiss [1970], [1971a], [1971b], [1972a], [1972b], Kiefer and Weiss [1971], [1974]; Hoel [1972]; Hoel and Sobel [1972]; Hoel, Sobel, Weiss [1972]; Nebenzahl and Sobel [1972]. During the period

1973-1980 a large number of additional papers appeared; all employed the measure of distance $\Delta_{i,j}$ (except Taheri and Young [1974] who used p_i/p_j). These papers are listed among our references. An excellent review of many of these proposed procedures (and others), with particular reference to adaptive sampling for clinical trials, is contained in Hoel, Sobel and Weiss [1975b]. A recent text by Büringer, Martin and Schriever [1980] gives an in depth comprehensive survey of these procedures (and many additional ones); it treats their derivation, performance characteristics, and uses, and provides extensive tables for their implementation.

Concurrently, the Bernoulli selection problem was studied employing the subset approach of Gupta [1956]. The early key papers using this approach are Gupta, Huyett and Sobel [1957] and Gupta and Sobel [1960]; an up-to-date summary of more recent results using the subset approach is contained in Gupta and Panchapakesan [1979], Section 13.2.

The problem of allocating observations among treatments when the total available number of observations is fixed (fixed patient horizon), with the objective of assigning a higher proportion of the total number of available observations to the population with the larger success probability has been studied in the medical context by Armitage [1960, 1975], Anscombe [1963], Colton [1963], Cornfield, Halperin and Greenhouse [1969], Zelen [1969], and Canner [1970], among others. For comments concerning this formulation of the problem see Sobel and Weiss [1972b].

A somewhat related class of procedures directed toward solutions of the so-called 2-armed (or multi-armed) bandit problem was investigated by many research workers: Robbins [1952], [1956]; Bradt, Johnson and Karlin [1956], Isbell [1959], Feldman [1962], Smith and Pyke [1965], Fabius and van Zwet [1970], Berry [1972], [1978], and Rodman [1978], among others. These papers are

not concerned with the Bernoulli selection problem, but rather focus on minimizing or maximizing appropriate objective functions, the principal tool used being dynamic programming.

2. The k-population Bernoulli selection problem

2.1 Earlier approaches

Before we describe our objectives and approach, it will be helpful to sketch the chronological development of certain statistical aspects of the Bernoulli selection problem. It perhaps is of historical interest to note that the Sobel-Huyett [1957] and Gupta-Huyett-Sobel [1957] papers made no reference to the potential applicability of their procedures to the drug selection problem or to clinical trials. Such a reference appears first in Paulson [1967] (although Armitage [1960, 1975], Anscombe [1963] and Colton [1963] had earlier considered such applications). Sobel and Weiss [1970] treated the special case $k = 2$, and emphasized the desirability of minimizing the number of patients on the poorer treatment. With this objective in mind they studied the performance of the play-the-winner (PW) sampling rule (introduced earlier by Robbins [1952], [1956], and proposed specifically for clinical trials by Zelen [1969]). The two procedures studied by Sobel-Weiss [1970] employed PW and VT sampling rules, the latter having been proposed earlier by Bechhofer, Kiefer and Sobel [1968] (B-K-S); both procedures suffered from the fact that the expected total number of observations ($E\{N\}$) required to terminate experimentation approached infinity both for PW and VT as either $p_{[1]} \to 1$ or $p_{[2]} \to 0$. To overcome this problem for VT for $k = 2$, Kiefer and Weiss [1971] suggested a truncated version of the B-K-S VT - procedure, and permitted the possibility of a third terminal decision, i.e., "The two populations are essentially the same." (Later, Kiefer and Weiss [1974] proposed an analogous truncated version of the

Sobel-Weiss PW sampling rule.) The procedures of Sobel and Weiss [1971b] for $k = 2$ and [1972a] for $k \geq 3$ which employed PW sampling and a stopping rule based on inverse sampling also were vulnerable to $p_{[k]} \to 0$ since then $E\{N\} \to \infty$.

The Sobel-Weiss [1972a] procedure was the first (after Paulson [1967], [1969]) to consider the case $k \geq 3$ for the distance measure $\Delta_{i,j}$. Although most investigators studied only the $k = 2$ case, Hoel and Sobel [1972], Sobel and Weiss [1972b], Hoel, Sobel and Weiss [1975a], and Schriever [1978/79] considered the $k \geq 3$ case. All restricted consideration to the goal of selecting the "best" population.

Further work on closed procedures for $k = 2$ was carried out by Hoel [1972], Nebenzahl and Sobel [1972], Berry and Sobel [1973], Fushimi [1973], Kiefer and Weiss [1974], Simon, Weiss and Hoel [1975], Schriever [1979] and Tamhane [1981]. Bofinger [1978] and Schriever [1978/79] appear to be the only authors to have considered closed procedures for $k \geq 3$. Most of these procedures employed some variant of PW sampling rules designed to minimize $E\{N\}$ and/or $E\{N_{(1)}\}$, the expected number of observations taken from the population associated with $p_{[1]}$.

2.2 Our approach

In this paper we have limited consideration to closed procedures, i.e., procedures for which the total number of observations taken from any of the $k \geq 2$ populations is a bounded random variable. We are disenchanted with open procedures because we believe that they are of little practical use. (This, of course, is a criticism of all of the ranking and identification procedures described in Bechhofer, Kiefer and Sobel [1968], and in a hypothesis testing or acceptance sampling context of the Wald sequential probability ratio test.) Even if $E\{N\}$ is "small" relative to the kn

required by the best competing single stage procedure, the distribution of N is usually highly skewed to the right, and hence "large" values of N occur with positive (albeit small) probabilities. This fact discourages the use of such procedures.

Our reference point is the single-stage procedure of Sobel and Huyett [1957] which takes exactly n observations from each of the $k \geq 2$ populations. We were able to characterize a class of closed sequential procedures which achieve the same probability of a correct selection as does the Sobel-Huyett (S-H) procedure, uniformly in $\underset{\sim}{p}$. Within this class we have found adaptive procedures which are uniformly in $\underset{\sim}{p}$ superior in terms of $E\{N\}$ to the S-H procedure. For $k = 2$ our procedure is optimal within a certain class.

Our closed sequential procedures for $k \geq 2$ are applicable to a broad class of general ranking and selection goals such as the one described in equation (6) of Bechhofer [1954], namely, "To select the k_t "best" populations, the k_{t-1} "second best" populations, etc., and finally the k_1 "worst" populations." Here $k_1, k_2, \ldots, k_t$ $(t \leq k)$ are positive integers such that $\sum_{i=1}^{t} k_i = k$. To illustrate our procedure we consider in Section 3 the case $t = 2$, $k_1 = k - s$, $k_2 = s$ $(1 \leq s \leq k - 1)$ which we call Goal I, and in Section 4 the case $t = s + 1$, $k_1 = k - s$, $k_2 = k_3 = \ldots = k_{s+1} = 1$ $(1 \leq s \leq k - 1)$ which we call Goal II. Other goals not given by (6) in Bechhofer [1954] can be handled similarly.

The main difference between our present formulation of the problem and that adopted in all of the previous papers in this category is that the so-called "least-favorable configuration" of the p-values (which plays a central role when designing an experiment using the indifference-zone approach) is of no concern to us. Our interest is focused on the probability of

achieving a correct selection for a given n for the particular goal considered, and in accomplishing this objective with minimum cost (e.g., minimum $E\{N\}$ needed to achieve a correct selection). A special virtue of all of our procedures for $k \geq 2$ is that no special tables of constants are necessary to carry out the procedures, and the procedures are very easy to implement.

We assume throughout that the response (success or failure) of an experiment is known sufficiently soon that it can influence the choice of population for the next experiment. This condition is not met in most clinical trials (although it often can be realized in testing in the physical sciences). Even if this condition is not met the procedures can sometimes be used to advantage. (See Remark 5.3.) Also, modifications of the procedure can be made to good effect if the responses are delayed.

3. Single-stage procedures

In this section we consider single-stage procedures for the Goal I and Goal II Bernoulli selection problems. Let $S_i(F_i)$ denote a "success" ("failure") from Π_i $(1 \leq i \leq k)$. If n observations are taken from Π_i, let $y_{i,n}$ denote the number of successes yielded by Π_i $(1 \leq i \leq k)$.

3.1 Single-stage procedure for Goal I

PROCEDURE FOR GOAL I (Selecting the s $(1 \leq s \leq k-1)$ "best" of k populations without regard to order):

Sampling rule (R_{SS}): Take n observations from each of the k populations. (3.1a)

Terminal decision rule (T_{SS}): Compute $y_{i,n}$ $(1 \leq i \leq k)$.

Let A_1, $A_2 \subset A = \{1,2,\ldots,k\}$ denote two disjoint sets of order s and $k-s$, respectively, such that

$$y_{i_1,n} \geq y_{i_2,n} \tag{3.1b}$$

for all $i_1 \in A_1$ and for all $i_2 \in A_2$. If there are r sets $A^{(i)} = \{A_1, A_2\}$ $(1 \leq i \leq r)$ satisfying (3.1b), then select one of them at random and announce for the selected set that A_1, A_2 are associated with $\{p_{[k]}, p_{[k-1]},\ldots,p_{[k-s+1]}\}$ and $\{p_{[k-s]},\ldots,p_{[1]}\}$, respectively.

3.2 Single-stage procedure for Goal II

PROCEDURE FOR GOAL II (Selecting the s $(1 \leq s \leq k-1)$ "best" of k populations with regard to order):

Sampling rule (R_{SS}): Take n observations from each of the k populations. (3.2a)

Terminal decision rule (T_{SS}): Compute $y_{i,n}$ $(1 \leq i \leq k)$.

Let $A_1, A_2,\ldots,A_{s+1} \subset A = \{1,2,\ldots,k\}$ denote $s+1$ disjoint sets, $A_1,\ldots,A_s$ of order one, A_{s+1} of order $k-s$, such that

$$y_{i_j,n} \geq y_{i_{j+1},n} \quad (1 \leq j \leq s) \tag{3.2b}$$

for $i_j \in A_j$ $(1 \leq j \leq s)$ and for all $i_{s+1} \in A_{s+1}$. If there are r sets $A^{(i)} = \{A_1,A_2,\ldots,A_{s+1}\}$ $(1 \leq i \leq r)$ satisfying (3.2b), then select one of them at random and announce for the selected set that $A_1,A_2,\ldots,A_s$ and A_{s+1} are associated with $p_{[k]}, p_{[k-1]},\ldots,p_{[k-s+1]}$ and $\{p_{[k-s]},\ldots,p_{[1]}\}$, respectively.

Example 3.1: $(k = 5, s = 3, n = 3)$

Π_1	Π_2	Π_3	Π_4	Π_5
S_1	S_2	S_3	S_4	F_5
S_1	S_2	S_3	S_4	F_5
S_1	S_2	F_3	F_4	F_5

Then

$$A^{(1)} = \{\{1\}, \{2\}, \{3\}, \{4,5\}\}$$
$$A^{(2)} \quad \{\{1\}, \{2\}, \{4\}, \{3,5\}\}$$
$$A^{(3)} = \{\{2\}, \{1\}, \{3\}, \{4,5\}\}$$
$$A^{(4)} = \{\{2\}, \{1\}, \{4\}, \{3,5\}\}.$$

Hence, select one of $A^{(i)}$ $(1 \leq i \leq 4)$ at random.

Remark 3.1: The single-stage procedures given in this section for Goals I and II coincide for $s = 1$. The case $s = 1$ was studied in detail by Sobel and Huyett [1957]. In that paper the common sample size n was chosen to guarantee certain indifference-zone probability requirements (as in Bechhofer [1954]) given by their equations (5) and (13).

4. A class of sequential procedures

We now propose a class of sequential procedures for the Goal I and Goal II Bernoulli selection problems. Let S_i^m (F_i^m) denote a success (failure) from Π_i at stage m $(1 \leq i \leq k, 1 \leq m \leq kn)$. Let $n_{i,m}$ denote the total number of observations taken from Π_i through stage m, and let $z_{i,m}$ denote the total number of successes yielded by Π_i through stage m $(1 \leq i \leq k, 1 \leq m \leq kn)$.

Theorem 5.1 (in Section 5.1) relates to a class of sequential selection

procedures which employs a very general class of sampling rules, and a particular stopping and terminal decision rule specific to the goal (Goal I or Goal II) under consideration.

Throughout the remainder of this paper we shall let R denote an arbitrary sampling rule which takes no more than n observations from any of the k populations. The basis for specifying n (e.g., to guarantee an indifference-zone probability requirement as in Sobel and Huyett or because of availability of observations or because of other economic considerations) is of no concern to us here.

4.1 A class of sequential procedures for Goal I

PROCEDURE FOR GOAL I (Selecting the s $(1 \leq s \leq k - 1)$ "best" of k populations without regard to order):

Sampling rule (R): Arbitrary, the only restriction being that at most n observations can be taken from any of the k populations. Thus, e.g., one-at-a-time sampling, play-the-winner sampling, vector-at-a-time sampling, or multistage sampling can be used. (4.1a)

Stopping rule (S^*): Stop sampling at the first stage m at which there exist two disjoint sets $A_1, A_2 \subset A = \{1,2,\ldots,k\}$ with A_1 of order s and A_2 of order $k - s$, such that

$$z_{i_1,m} \geq z_{i_2,m} + n - n_{i_2,m} \tag{4.1b}$$

for all $i_1 \in A_1$ and for all $i_2 \in A_2$.

Terminal decision rule (T^*): If r sets $A^{(i)} = \{A_1, A_2\}$ $(1 \leq i \leq r)$ satisfy (4.1b), then select one of them at random and announce for the selected set that A_1 and A_2 are associated with (4.1c)

$\{p_{[k]}, p_{[k-1]}, \cdots, p_{[k-s+1]}\}$ and $\{p_{[k-s]}, \cdots, p_{[1]}\}$, respectively.

Examples 4.1: For (k = 3, s = 1, n = 1), stop if

Π_1	Π_2	Π_3		Π_1	Π_2	Π_3
		S_3^1		F_1^1	F_2^2	

Then $A^{(1)} = \{\{3\}, \{1,2\}\}$. Hence, select $A^{(1)}$.

Examples 4.2: For (k = 3, s = 2, n = 1), stop if

Π_1	Π_2	Π_3		Π_1	Π_2	Π_3
		F_3^1		S_1^1		F_3^2

Then $A^{(1)} = \{\{1,2\}, \{3\}\}$. Hence, select $A^{(1)}$.

Example 4.3: For (k = 5, s = 2, n = 3), stop if

Π_1	Π_2	Π_3	Π_4	Π_5
S_1^{10}	S_2^7	S_3^4	F_4^3	S_5^1
S_1^{11}	S_2^8	S_3^5		F_5^2
	F_2^9	F_3^6		

Then $A^{(1)} = \{\{1,2\}, \{3,4,5\}\}$ and $A^{(2)} = \{\{1,3\}, \{2,4,5\}\}$.
Hence, select one of $A^{(i)}$ (i = 1,2) at random.

4.2 A class of sequential procedures for Goal II

PROCEDURE FOR GOAL II (Selecting the s $(1 \leq s \leq k-1)$ "best" of k populations with regard to order):

Sampling rule (R): Arbitrary, the only restriction being that at most n observations can be taken from any of the k populations. (4.2a)

Stopping rule (S^*): Stop sampling at the first stage m at which there exist $s+1$ disjoint sets $A_1, A_2, \ldots, A_s, A_{s+1} \subset A = \{1,2,\ldots,k\}$ with $A_1, \ldots, A_s$ of order one, A_{s+1} of order $k-s$, such that

$$z_{i_j,m} \geq z_{i_{j+1},m} + n - n_{i_{j+1},m} \quad (1 \leq j \leq s) \tag{4.2b}$$

for $i_j \in A_j$ $(1 \leq j \leq s)$ and for all $i_{s+1} \in A_{s+1}$.

Terminal decision rule (T^*): If r sets $A^{(i)} = \{A_1, A_2, \ldots, A_{s+1}\}$ $(1 \leq i \leq r)$ satisfy (4.2b) then select one of them at random and announce for the selected set that $A_1, A_2, \ldots, A_s$ and A_{s+1} are associated with $p_{[k]}, p_{[k-1]}, \ldots, p_{[k-s+1]}$ and $\{p_{[k-s]}, \ldots, p_{[1]}\}$, respectively. (4.2c)

Example 4.4: For $(k = 5, s = 3, n = 3)$, stop if

Π_1	Π_2	Π_3	Π_4	Π_5
S_1^6	S_2^4	F_3^3	F_4^2	F_5^1
S_1^7	F_2^5	S_3^{13}	S_4^{11}	F_5^{10}
S_1^8	F_2^9		F_4^{12}	

Then $A^{(1)} = \{\{1\}, \{3\}, \{2\}, \{4,5\}\}$ and $A^{(2)} = \{\{1\}, \{3\}, \{4\}, \{2,5\}\}$. Hence, select one of $A^{(i)}$ $(i = 1,2)$ at random.

5. Comparison of some performance characteristics of the single-stage and sequential procedures.

In this section we compare the probability of a correct selection achieved by our class of sequential procedures with that of the corresponding single-stage procedures. We do the same for the total number of observations required to terminate experimentation for the sequential procedures and the total sample size required by the corresponding single-stage procedures.

5.1 Probability of a correct selection

If two or more populations have a common p-value, assume that the populations are tagged in such a way that the ordering of the k populations is unique. Then *a correct selection (CS) for Goal I is achieved if the* selected sets A_1, A_2 are associated with $\{p_{[k]}, p_{[k-1]}, \ldots, p_{[k-s+1]}\}$, $\{p_{[k-s]}, \ldots, p_{[1]}\}$, respectively; analogously, a correct selection for Goal II is achieved if the selected sets $A_1, A_2, \ldots, A_{s+1}$ are associated with $\{p_{[k]}\}$, $\{p_{[k-1]}\}, \ldots, \{p_{[k-s+1]}\}$, $\{p_{[k-s]}, \ldots, p_{[1]}\}$, respectively.

We now state our first key theorem relating the P{CS} achieved by our sequential procedures and the P{CS} achieved by the corresponding single-stage procedures.

Theorem 5.1:

$P_I\{CS \mid (R_{SS}, T_{SS})\} \equiv P_I\{CS \mid (R, S^*, T^*)\}$ uniformly in $\underset{\sim}{p} = (p_1, \ldots, p_k)$ for Goal I, and analogously for Goal II.

Proof: See Appendix A.

Remark 5.1: Note that if the weak inequality in (4.1b) and (4.2b) were replaced by a strict inequality, the associated stopping rules would involve curtailment of the sampling process. Then the conclusion of Theorem 5.1 would be obvious since the resulting sequential procedure and the single-stage procedure always lead to the same terminal decision. However, such is not the case when the weak inequality is used. For example, for $k \geq 2$, $s = 1$, $n \geq 1$, we see that (4.1b) calls for stopping if (say) the sequence S_i^j $(1 \leq j \leq n)$ were obtained for any i $(1 \leq i \leq k)$ in which situation (4.1c) would select Π_i. However, for that same initial sequence curtailed sampling would require that at least one more observation be taken from all populations Π_j $(j \neq i, 1 \leq j \leq k)$; if these additional observations were such that a total of $r - 1$ additional populations also yielded n S's, then the curtailment terminal decision rule would select one of these r n-success populations at random (which is what the single-stage procedure would do). Thus (4.1b) and (4.1c) permit earlier stopping than under curtailment, but sometimes may lead to a different terminal decision than under curtailment.

Remark 5.2: If sampling continues beyond the stage called for by (4.1b) for Goal I or (4.2b) for Goal II then the P{CS} is not increased (provided that the total number of observations taken from any population is at most n). VT sampling for the Bernoulli selection problem always requires at least as large a total number of observations as would be required by a one-at-a-time sampling rule (and, in fact, often a very much larger total) to achieve the same P{CS} for a given data set. Thus, unless VT sampling is used for (say) "blocking" purposes for the Bernoulli, it should ordinarily be avoided. For an example of the latter situation see Tamhane [1980].

Remark 5.3: In some areas of application, e.g., in certain types of clinical trials and in reliability-life studies, the experiments may be started at different times, and the outcomes (successes or failures) from the k populations may be staggered or spaced over time. This might be the case in experiments which are designed in single-stages as with Sobel-Huyett and for which n observations are to be taken from each of the k populations. In such situations for (say) Goal I, the stopping rule (4.1b) and the terminal decision rule (4.1c) can be applied as each success or failure is recorded. Then (4.1b) permits the possibility of an early terminal decision although successes and failures will continue to be recorded as they occur after that point. These later observations make it possible to estimate the p_i $(1 \leq i \leq k)$ more precisely. They may lead to a different terminal decision, but they will not increase the probability of a correct selection.

Remark 5.4: If the common sample size n of the Sobel-Huyett single-stage procedure was chosen to guarantee the indifference-zone probability requirements given by their equations (5) or (13), then à fortiori our class of sequential procedures for $s = 1$ guarantees these same probability requirements. Although Sobel-Huyett did not consider Goal I or Goal II for $s > 1$, an analogous result would hold for those goals as well.

Remark 5.5: For large n the normal approximation to the binomial distribution can be used (as in Sobel-Huyett) to obtain an excellent approximation to the P{CS} achieved by the single-stage procedure for Goal I or Goal II for specified s and given $\underset{\sim}{p}$. This computed P{CS} thus holds for our general class of sequential procedures for the same specified s and given $\underset{\sim}{p}$.

Remark 5.6: A single-stage procedure for selecting the multinomial event which has the largest probability is described in Bechhofer, Elmaghraby and

Morse [1959]; only the case $s = 1$ was considered. Alam and Thompson [1972] proposed a single-stage procedure for the case $s = k - 1$. The sequential procedures employing vector-at-a-time (VT) sampling and (S^*, T^*) for the Bernoulli selection problem given for Goal I $(1 \leq s \leq k - 1)$ and Goal II $(1 \leq s \leq k - 1)$ in Section 4 of the present paper are directly applicable to the multinomial selection problem (with obvious interpretations of notation).

A sequential procedure employing multinomial VT sampling with curtailment was proposed for the multinomial selection problem $(s = 1)$ by Gibbons, Olkin and Sobel [1977], pp. 178-183. Our procedure improves on the G-O-S procedure in that it achieves the same $P\{CS\}$ uniformly in $\underset{\sim}{p}$ as does theirs (and the single-stage procedure), but our procedure requires at most as many (and usually less) vector-stages to terminate sampling. These results with accompanying computations are contained in Bechhofer and Kulkarni [1981].

5.2 Total number of observations to terminate sampling

In Sections 4.1 and 4.2 we described a class of sequential procedures for Goals I and II, respectively. For each the sampling rule is arbitrary, the only restrictions being that the rule adopted take no more than n observations from any of the k populations, and that it be used in conjunction with (4.1b) and (4.1c) for Goal I or (4.2b) and (4.2c) for Goal II. If we denote by N the random total number of observations that have been taken from the k populations when sampling stops, then it can be shown that using (4.1b) and (4.1c) for Goal I we have

$$\min\{sn, (k-s)n\} \leq N \leq kn, \tag{5.1}$$

or using (4.2b) and (4.2c) for Goal II we have

$$sn \leq N \leq kn; \tag{5.2}$$

if an arbitrary one-at-a-time sampling rule is used, then

$$N \leq kn - 1 \tag{5.3}$$

for both Goal I and Goal II. Examples 4.1 and 4.2 show that the lower bound in (5.1) and the upper bound in (5.3) can be achieved for appropriate sampling rules and outcomes. That the procedures are closed is of particular practical importance.

For either Goal I or Goal II with given (k, s, n), the distribution of N and hence $E\{N|\underset{\sim}{p}\}$ (and other related performance characteristics of the sequential procedure) depend on $\underset{\sim}{p}$ and the specific sampling rule that is used. In Sections 6 and 7 we propose a particular sampling rule which has highly desirable properties when used in conjunction with (S^*,T^*).

6. An optimal sequential procedure for k = 2

In this section and the next we continue to restrict attention to arbitrary sampling rules R which take no more than n observations from any of the k populations, and which are used in conjunction with (4.1b) and (4.1c) for Goal I or (4.2b) and (4.2c) for Goal II. We seek sampling rules within this class which have desirable properties. Indeed we have been successful in constructing an optimal rule for k = 2 for several definitions of optimality which are of considerable practical importance. Our results are summarized in Theorems 6.1, 6.2 and 6.3, below.

In the sequel we let $N_{(i)}$ ($N^S_{(i)}$, $N^F_{(i)}$) denote the random number of

observations (successes, failures) that have been taken from the population with parameter $p_{[i]}$ $(1 \leq i \leq k)$ when sampling stops. Also let $N^S = \sum_{i=1}^{k} N^S_{(i)}$ and $N^F = \sum_{i=1}^{k} N^F_{(i)}$. Then $N = \sum_{i=1}^{k} N_{(i)} = N^S + N^F$. We are particularly interested in $E\{N\}$. However, also of concern is $E\{\sum_{i=1}^{k-1} N_{(i)}\}$, the expected total number of observations taken from the "inferior" populations, i.e., those having the smaller p-values; this quantity is especially important in clinical trials where ethical considerations play an important role, and p_i is the probability of a "success" using treatment i $(1 \leq i \leq k)$. (See Hoel, Sobel and Weiss [1975b].) For the same reason $E\{N^F\}$ is important. In each case we seek to make $E\{N\}$, $E\{\sum_{i=1}^{k-1} N_{(i)}\}$ and $E\{N^F\}$ as small as possible. It is obvious that for these objectives it is sufficient to restrict attention to one-at-a-time sampling rules.

6.1 Minimization of $E\{N\}$

We use the following notation for $k = 2$. For a state $(z_{1,m}, n_{1,m}; z_{2,m}, n_{2,m})$ which does not satisfy (4.1b), let $D_m = D_m(z_{1,m}, n_{1,m}; z_{2,m}, n_{2,m})$ denote the sampling decision at stage m $(m = 0, 1, \ldots, 2n-1)$. $D_m = i$ means that at stage m the next observation is to be taken from Π_i $(i = 1,2)$; $D_m = (1,2)$ means that at stage m the next observation is to be taken at random from Π_1 or Π_2.

Sampling rule (R^*):

$$D_m = \begin{cases} 1 \text{ if } n_{1,m} - z_{1,m} < n_{2,m} - z_{2,m} \\ \quad \text{or} \\ \quad n_{1,m} - z_{1,m} = n_{2,m} - z_{2,m} \text{ and } z_{1,m} > z_{2,m}, \\ 2 \text{ if } n_{1,m} - z_{1,m} > n_{2,m} - z_{2,m} \\ \quad \text{or} \\ \quad n_{1,m} - z_{1,m} = n_{2,m} - z_{2,m} \text{ and } z_{1,m} < z_{2,m}, \\ (1,2) \text{ if } n_{1,m} - z_{1,m} = n_{2,m} - z_{2,m} \text{ and } z_{1,m} = z_{2,m}. \end{cases} \tag{6.1}$$

Theorem 6.1: Among all sampling rules R used in conjunction with (S^*,T^*) for $k = 2$, R^* minimizes $E\{N|(p_1,p_2)\}$ for $p_1 + p_2 > 1$. The conjugate sampling rule $\overline{R}^*$ (in which $n_{i,m} - z_{i,m}$ and $z_{i,m}$ in R^* are replaced by $z_{i,m}$ and $n_{i,m} - z_{i,m}$, respectively, for $i = 1,2$) minimizes $E\{N|(p_1,p_2)\}$ for $p_1 + p_2 < 1$. Both R^* and $\overline{R}^*$ minimize $E\{N|(p_1,p_2)\}$ for $p_1 + p_2 = 1$, and (for symmetry) one can choose between them with probability (1/2, 1/2).

Proof: The proof of Theorem 6.1 is quite long, and therefore is not given here. It is given in detail in Kulkarni [1981] along with the proofs of Theorems 6.2 and 6.3 which are stated below. These proofs will be published elsewhere.

Remark 6.1: Note that R^* and $\overline{R}^*$ are stationary, i.e., the rules are independent of n. (Contrast this result with the one described in Remark 6.5.)

Example 6.1: To illustrate the sequential procedure which employs (R^*,S^*,T^*) for $k = 2$, $s = 1$ and $n = 7$ we give the following stopping sequence:

	Π_1	Π_2
Cycle 1	F_1^1	S_2^2
		F_2^3
Cycle 2	S_1^7	S_2^4
	F_1^8	S_2^5
		F_2^6
Cycle 3 is		S_2^9
truncated by S^*		S_2^{10}

Π_2 is selected after S_2^{10}. Note that we regard the sampling as proceeding in cycles; within each cycle (except perhaps the last one) the outcomes from each population are a sequence of successes followed by a single failure. Here the last cycle is truncated by S^*.

Remark 6.2: Note from S_2^4 of Example 6.1 that R^* is not a PW sampling rule. (See Robbins [1956], Zelen [1969], Sobel and Weiss [1970].) R^* is PW within a cycle, but may not be PW as sampling progresses from

one cycle to the next. Play-the-loser for $\bar{R}^*$ corresponds to PW for R^*. Most of the sequential procedures proposed in the literature for the Bernoulli selection problem employ a PW sampling rule.

Remark 6.3: If the experimenter knows that $p_1 + p_2 > 1$ ($p_1 + p_2 < 1$) then he presumably would use R^* ($\bar{R}^*$). However, lacking such knowledge he may be prepared to assume that (p_1,p_2) represent the outcome of a random sample of size two taken from a Beta distribution $B(a,b)$: $[\Gamma(a+b)/\Gamma(a)\,\Gamma(b)]\, x^{a-1}\,(1-x)^{b-1}$, $0 \leq x \leq 1$, $a,b > 0$. Since $P\{p > 1/2 \mid (a,b)\} > 1/2$ for $a > b$, he may wish to replace the assumption $p_1 + p_2 > 1$ ($p_1 + p_2 < 1$) by the assumption $a > b$ ($a < b$), and chose (a,b) accordingly to model his assessment of the particular situation under study. Then he can use the following empirical sampling rule R_E^* in conjunction with (S^*,T^*).

Empirical sampling rule (R_E^*):

At stage m estimate p_i $(i = 1,2)$ by $\hat{p}_{i,m} = (z_{i,m} + a)/(n_{i,m} + a + b)$.

$$\begin{aligned} &\text{If} \quad \hat{p}_{1,m} + \hat{p}_{2,m} > 1 \quad \text{use} \quad R^*, \\ &\hat{p}_{1,m} + \hat{p}_{2,m} < 1 \quad \text{use} \quad \bar{R}^*, \\ &\hat{p}_{1,m} + \hat{p}_{2,m} = 1 \quad \text{use either} \quad R^* \quad \text{or} \quad \bar{R}^*. \end{aligned} \tag{6.2}$$

Remark 6.4: Based on limited calculation for selected (a,b) and $n \leq 10$ it appears that the $E\{N\}$ - values obtained for R_E^* and the optimal Bayes sampling rule are very close. Here the expectation is taken w.r.t. the prior Beta density.

Note: Since $\hat{p}_{i,m} \to p_i$ $(i = 1,2)$ for $m \to \infty$, an error in the choice of the particular (a,b) will have little effect on the performance of (R^*_L, S^*, T^*) when n is large.

6.2 Minimization of $E\{N_{(1)}\}$.

We had mentioned that in clinical trials it would be desirable to minimize the expected total number of observations taken from the populations *with small p-values*. Our next theorem addresses that issue for $k = 2$.

Theorem 6.2: Among all sampling rules R used in conjunction with (S^*, T^*) for $k = 2$, R^* minimizes $E\{N_{(1)} | (p_1, p_2)\}$ for $p_{[2]} > 1/2$.

Remark 6.5: If $p_{[2]} < 1/2$ it is not possible to find a stationary sampling rule which when used in conjunction with (S^*, T^*) for $k = 2$ will minimize $E\{N_{(1)} | (p_1, p_2)\}$ for all (p_1, p_2), as is illustrated by the following example.

Example 6.2: Let $k = 2$ and suppose that $p_{[1]} = 0.085$, $p_{[2]} = 0.250$. Suppose that $z_{1,1} = 1$, $n_{1,1} = 1$, $z_{2,1} = 0$, $n_{2,1} = 0$. Using dynamic programming (DP) it can be shown that at stage 1 the sampling rule that minimizes $E\{N_{(1)}\}$ for this particular pair $(p_{[1]}, p_{[2]})$ with $p_{[2]} < 1/2$ and outcome $(1,1;0,0)$ is "Select the next observation from Π_2 if $n = 2$; select the next observation from Π_1 if $n = 3$." Thus the optimal sampling rule depends on n, and hence is not stationary.

6.3 Minimization of $E\{N^F\}$.

In clinical trials it is undesirable to obtain failures with any of the treatments employed in the trial. Our next theorems relate to that problem for $k = 2$.

Theorem 6.3A: Among all sampling rules R used in conjunction with (S^*,T^*) for $k = 2$, R^* minimizes $E\{N^F|(p_1,p_2)\}$ for $p_1 + p_2 > 1$.

Theorem 6.3B: Among all sampling rules R used in conjunction with (S^*,T^*) for $k = 2$, $\bar{R}^*$ minimizes $E\{N^F|(p_1,p_2)\}$ for $p_{[2]} < 1/2$.

Remark 6.7: If $p_1 + p_2 < 1$ and $p_{[2]} > 1/2$ there exist points (p_1,p_2) such that neither R^* nor $\bar{R}^*$ when used in conjunction with (S^*,T^*) for $k = 2$ will minimize $E\{N^F|(p_1,p_2)\}$, as is illustrated by the following example.

Example 6.3: Let $k = 2$ and suppose that $n = 2$ and $p_{[1]} = 0.10$, $p_{[2]} = 0.55$. Suppose that $z_{1,1} = 1$, $n_{1,1} = 1$, $z_{2,1} = 0$, $n_{2,1} = 0$; using DP it can be shown that at stage 1 the R that minimizes $E\{N^F\}$ for this particular pair $(p_{[1]},p_{[2]})$ and outcome is R^*. Suppose now that $z_{1,1} = 0$, $n_{1,1} = 1$, $z_{2,1} = 0$, $n_{2,1} = 0$; using DP it can be shown that at stage 1 the R that minimizes $E\{N^F\}$ for the same particular pair $(p_{[1]},p_{[2]})$, but different outcome is $\bar{R}^*$.

Theorems 5.1, 6.1, 6.2 and 6.3 summarize four highly desirable properties of R^* when used in conjunction with (S^*,T^*) for the two-population Bernoulli selection problem. In Section 7 we consider sampling rules for the k-population ($k \geq 3$) problem.

7. A proposed sampling rule for Goals I and II for $k > 2$

In this section we propose a natural generalization to $k > 2$ of the sampling rule R^*. This generalized R^* (which we still will refer to as R^* since it reduces to R^* when $k = 2$) when used in conjunction with (S^*,T^*) is thus a member of the class of sequential procedures described in

Section 4; hence Theorem 5.1 applies. We describe some of its desirable properties in Section 7.1, and conjecture an optimal property in Section 7.2.

Generalized sampling rule (R^*):

At stage m $(0 \leq m < kn-1)$, if sampling has not stopped, take the next observation from the population which has the smallest number of failures among all Π_i for which $n_{i,m} < n$ $(1 \leq i \leq k)$. If there is a tie among such equal-number-of-failure populations, take the next observation from that one of them which has the largest number of successes. If there is a further tie among such equal-number-of-success populations, select one of them at random and take the next observation from it.

Remark 7.1: We can think of the sampling rule R^* as proceeding in cycles. Before the start of sampling the populations are arranged in random order, say, $\Pi_1', \Pi_2', \ldots, \Pi_k'$. The first cycle is started by taking one observation at a time from Π_1' until a failure is obtained. Then observations are taken one-at-a-time from Π_2' until a failure is obtained. This process is continued until every population has produced a sequence of successes followed by a single failure. Then the first cycle is complete, and every population has produced exactly one failure (unless truncation has occurred during the cycle). Cycle i is started by taking observations from the population which has the largest cumulative number of successes through cycle $i - 1$ $(1 \leq i \leq c)$ where c is the random total number of cycles until the termination of sampling. That population is sampled until a failure is obtained. The cycle is continued by sampling from the population which has the second largest cumulative number of successes through cycle $i - 1$, and sampling from that population is continued until a failure is obtained. This process is continued until in cycle i every population has produced a sequence of successes

followed by a single failure. Then the ith cycle is complete, and every population has produced a cumulative number of exactly i failures (unless truncation has occurred during the ith cycle). If within a cycle two or more populations which have not yet been sampled in that cycle have the same cumulative number of successes through cycle i - 1, then they are sampled in random order.

Remark 7.2: Sampling rule R^* had been proposed earlier for $k \geq 2$ by a referee of Sobel and Weiss [1972a]; see the sampling rule $\hat{R}_I$ referred to on pp. 1809 and 1824 of their paper. This sampling rule was to be used in conjunction with a stopping rule based on inverse sampling. However, as noted above, our motivation for proposing R^* is that it is a natural generalization to $k > 2$ of the sampling rule of (6.1).

Example 7.1: To illustrate the sequential procedure which employs (R^*, S^*, T^*) for $k = 3$, $s = 1$, $n = 8$ we give the following stopping sequence:

	Π_1	Π_2	Π_3
	S_1^5	S_2^1	F_3^4
Cycle 1	F_1^6	S_2^2	
		F_2^3	
	S_1^9	S_2^7	S_3^{13}
Cycle 2	S_1^{10}	F_2^8	F_3^{14}
	S_1^{11}		
	F_1^{12}		
Cycle 3 is	S_1^{15}		
truncated by S^*	S_1^{16}		

Π_1 is selected after S_1^{16}.

Remark 7.3: The stopping sequences given in Examples 4.1 - 4.4 could have been obtained using (R^*,S^*,T^*).

Example 7.2: We illustrate Theorem 5.1 by calculating the exact P{CS} when (R_{SS},T_{SS}) and (R^*,S^*,T^*) are used for k = 3, s = 1, n = 1 (as in Examples 4.1).

Table 7.1

Outcomes leading to CS[1] for (R_{SS},T_{SS})	Probability of outcome and then CS[1]	Stopping sequences leading to CS[1] for (R^*,S^*,T^*)	Probability of stopping sequence and then CS[1]
(S_1,S_2,S_3)	$p_1\ p_2\ p_3\ \frac{1}{3}$	S_3^1	$\frac{1}{3}\ p_3$
(F_1,S_2,S_3)	$(1-p_1)p_2\ p_3\ \frac{1}{2}$	$F_1^1\ S_3^2$	$\frac{1}{3}\ (1-p_1)\frac{1}{2}\ p_3$
(S_1,F_2,S_3)	$p_1(1-p_2)p_3\ \frac{1}{2}$	$F_2^1\ S_3^2$	$\frac{1}{3}\ (1-p_2)\frac{1}{2}\ p_3$
(F_1,F_2,S_3)	$(1-p_1)(1-p_2)p_3$	$F_1^1\ F_2^2$	$\frac{1}{3}\ (1-p_1)\frac{1}{2}(1-p_2)$
(F_1,F_2,F_3)	$(1-p_1)(1-p_2)(1-p_3)\ \frac{1}{3}$	$F_2^1\ F_1^2$	$\frac{1}{3}\ (1-p_2)\frac{1}{2}(1-p_1)$

$$P\{CS\} = \frac{1}{3} - \frac{1}{3}(p_1+p_2-2\ p_3) + \frac{1}{6}[2\ p_1\ p_2 - (p_1+p_2)p_3]$$

[1] For simplicity of notation in this example we have assumed that $\max(p_1,p_2) < p_3$.

7.1 Some properties of the procedure (R^*,S^*,T^*)

The sequential procedure employing (R^*,S^*,T^*) has the following properties for $k \geq 2$:

a) $P\{CS|(R^*,S^*,T^*), \underset{\sim}{p}\} = P\{CS|(R_{SS},T_{SS}), \underset{\sim}{p}\}$ uniformly in $\underset{\sim}{p}$ for both Goal I and Goal II.

b) For Goal I

$$\min\{sn, (k-s)n\} \leq N \leq kn - 1,$$

and for Goal II

$$sn \leq N \leq kn - 1$$

for all $\underset{\sim}{p}$. These bounds on N can be achieved.

c) $\frac{1}{kn}(100) \leq \frac{kn - E\{N\}}{kn}(100) \leq \frac{k - s}{k}(100)$

for Goal II and analogously for Goal I for all $\underset{\sim}{p}$.

Here $(kn - E\{N\})100/kn$ is the percent saving in expected total number of observations if (R^*,S^*,T^*) is used in place of the corresponding single-stage procedure. This saving is <u>always</u> positive and can be very large for $p_{[1]} \to 1$.

d) $P\{N = sn|\underset{\sim}{p}\} \to 1$ for $p_{[1]} \to 1$ for Goals I and II,

$P\{N = kn - s|\underset{\sim}{p}\} \to 1$ for $p_{[k]} \to 0$ for Goal I,

$P\{N = kn-1|\underset{\sim}{p}\} \to 1$ for $p_{[k]} \to 0$ for Goal II.

e) Populations with small p-values tend to be sampled less frequently.

f) No special tables of constants are necessary to carry out (R^*,S^*,T^*) for $k \geq 2$, and it is very easy to implement.

We note from b) that using R^* instead of an arbitrary R with (S^*,T^*) yields a smaller upper bound $(kn - 1$ instead of $kn)$ for N. The fact that the sequential procedure employing (R^*,S^*,T^*) is <u>closed</u> increases its potential for use in real-life applications.

7.2 Some conjectures concerning the procedure (R^*,S^*,T^*)

For $k > 2$ we make a conjecture for (R^*,S^*,T^*) that is a generalization of Theorem 6.1. The conjecture is made only for the case $s = 1$; when $s > 1$ the situation is much more complicated, and it is very difficult to conjecture an optimal sampling rule even for a limited region of the parameter space. (See Remark 7.7.)

Conjecture 7.1: Among all sampling rules R used in conjunction with (S^*,T^*) for $k > 2$, $s = 1$, R^* minimizes $E\{N|\underset{\sim}{p}\}$ for $p_{[1]} + p_{[2]} > 1$.
Note: If this conjecture is true then for $s = k - 1$ generalized $\overline{R}^*$ (defined in the obvious way) minimizes $E\{N|\underset{\sim}{p}\}$ for $p_{[k]} + p_{[k-1]} < 1$.

Remark 7.4: Conjecture 7.1 was checked numerically for $k = 3$, $s = 1$, $n = 1(1)6$ over a fairly fine grid in the region $p_{[1]} + p_{[2]} > 1$, and was found to be true.

Remark 7.5: For $k = 3$, $s = 1$, $n = 1(1)6$ the condition $p_{[1]} + p_{[2]} > 1$ of Conjecture 7.1 is not necessary. For example, it was found by solving the dynamic programming equations on the computer that R^* minimizes $E\{N|\underset{\sim}{p}\}$ for $(p_{[1]},p_{[2]},p_{[3]}) = (0.35, 0.5, 0.7)$.

Remark 7.6: For $k = 3$, $s = 1$ it is not possible to find a stationary sampling rule that minimizes $E\{N|\underset{\sim}{p}\}$ for all $\underset{\sim}{p}$ such that $p_{[1]} + p_{[2]} < 1$ as is illustrated by the following example:

Example 7.1: ($k = 3$, $s = 1$, $n \geq 3$) Let $(p_{[1]},p_{[2]},p_{[3]}) = (0.4, 0.5, 0.6)$.

Consider the outcomes

Π_1	Π_2	Π_3
	S_2^4	S_3^1
	S_2^5	S_3^2
		F_3^3

For $n = 3$ the optimal sampling rule takes the next observation from Π_1 while R^* takes the next observation from Π_2. However, for $n = 4$ the optimal sampling rule takes the next observation from Π_2. Thus the optimal sampling rule is not stationary.

Remark 7.7: For $(k = 3, s = 2, n = 3)$, Goal I, the following example shows that R^* used with (S^*,T^*) does not minimize $E\{N|\underset{\sim}{p}\}$ for any $\underset{\sim}{p}$ such that $0 < p_{[1]} \leq p_{[3]} < 1$:

Example 7.2:

Π_1	Π_2	Π_3
S_1^3	F_2^2	F_3^1
S_1^4		

R^* takes the next observation from Π_1. However, the optimal sampling rule takes no more observations from Π_1, but chooses at random between Π_2 and Π_3 for the next observation. (A similar example can be constructed for

$k > 3$, $k > s > 1$.)

In Theorems 6.1 - 6.3 and Conjecture 7.1 we have limited consideration to a class of sampling rules which take at most n observations from each of the k populations, and which are used in conjunction with (S^*,T^*). We believe that the conclusions given in these theorems and conjecture hold for a broader class of stopping and terminal decision rules. Our belief is summarized in the following conjecture.

Conjecture 7.2: Among all sampling rules R used in conjunction with a stopping rule (S) and terminal decision rule (T) which achieve the same P{CS} as (R_{SS},T_{SS}) uniformly in $\underset{\sim}{p}$, (R^*,S^*,T^*) has the optimal properties described in Theorems 6.1 - 6.3 and Conjecture 7.1.

8. Performance of (R^*,S^*,T^*) relative to that of other Bernoulli selection procedures.

For the purpose of comparing the performance characteristics of (R^*,S^*,T^*) with those of some competing Bernoulli selection procedures, we have categorized the latter into two groups: those that achieve the same P{CS} as (R^*,S^*,T^*) uniformly in $\underset{\sim}{p}$, and those that do not.

8.1 Procedures having the same P{CS} as (R^*,S^*,T^*)

a) Hoel [1972] proposed a closed sequential procedure for $k = 2$ employing a PW sampling rule (R_{PW}). It can be shown that Hoel's procedure employs the stopping rule (S^*) and terminal decision rule (T^*). His stopping rule depends on a constant r which is actually the same for specified $\{\Delta^*,P^*\}$ as the n-value necessary to implement the Sobel-Huyett single-stage procedure (R_{SS},T_{SS}) for the same specification. Thus (R_{PW},S^*,T^*) achieves the same P{CS} uniformly in $\underset{\sim}{p}$ as do (R^*,S^*,T^*)

and (R_{SS},T_{SS}). Hoel gives tables of r-values for $\Delta^* = 0.1, 0.2, 0.3$ and $P^* = 0.90, 0.95, 0.99$. Using these $(r = n)$ - values for (R^*,S^*,T^*) and (R_{PW},S^*,T^*) we have found by exact calculations (for configurations with $p_{[1]} = p_{[2]}$ and $p_{[2]} = p_{[1]} + \Delta^*$) that $E\{N|\underset{\sim}{p};\ (R^*,S^*,T^*)\}$ $< (>)$ $E\{N|\underset{\sim}{p};(R_{PW},S^*,T^*)\}$ if $p_1 + p_2 > 1$ (< 1); however, $E\{N|\underset{\sim}{p};\ (\overline{R}^*,S^*,T^*)\} < <$ $E\{N|\underset{\sim}{p};(R_{PW},S^*,T^*)\}$ if $p_1 + p_2 < 1$. The $E\{N|R^*\}$ and $E\{N|\overline{R}^*\}$ inequalities for $p_1 + p_2 > 1$ and $p_1 + p_2 < 1$, respectively, follow, of course, from Theorem 6.1. Moreover, it follows from Theorem 6.2 that $E\{N_{(1)}|\underset{\sim}{p};(R^*,S^*,T^*)\} < E\{N_{(1)}|\underset{\sim}{p};(R_{PW},S^*,T^*)\}$ if $p_{[2]} > 1/2$; in fact, it appears from our computations that the inequality holds for all $p_{[2]} > p_{[1]}$. See also Pradhan and Sathe [1974].

b) Nebenzahl and Sobel [1972] proposed two sequential procedures, the first employing a vector-at-a-time sampling rule (R_{VT}) and the second employing a PW sampling rule (R_{PW}), both stopping when a fixed <u>total</u> number of observations n_T had been taken. They (N-S) chose n_T to guarantee the $\{\Delta^*,P^*\}$ indifference-zone probability requirement. N-S showed that for the same n_T both procedures achieved the same $P\{CS\}$ uniformly in $\underset{\sim}{p}$, and that the total number of observations taken from the population with $p_{[1]}$ was never more for R_{PW} than for R_{VT}. If n_T is even (odd) it can be shown that the $P\{CS\}$ using (R^*,S^*,T^*) with $n = n_T/2$ $((n_T+1)/2)$ is equal to that achieved by the N-S procedures with n_T, uniformly in $\underset{\sim}{p}$. Also, if n_T is even (odd) then $E\{N\}$ using (R^*,S^*,T^*) with $n = n_T/2$ $(n = (n_T+1)/2)$ is smaller (smaller except when $p_1 = p_2 = 0$) than that obtained using the N-S procedures with n_T, uniformly in $\underset{\sim}{p}$. See Büringer, Martin and Schriever [1980], p. 262, for further comments. Also see Pradhan and Sathe [1973], [1976].

c) Bofinger [1978] proposed a variant of a PW sampling rule with curtailment which is applicable to the Bernoulli selection problem for $k \geq 2$. She considered our Goal I, her "best" populations being the ones associated with $\{p_{[1]}, p_{[2]}, \ldots, p_{[k-s]}\}$. (Her PW corresponds to our play-the-loser.) Bofinger's procedure achieves the same P{CS} uniformly in $\underset{\sim}{p}$ as does (R_{SS}, T_{SS}) which uses the same n-value, and hence also as does (R, S^*, T^*). It would appear that generalized $(\bar{R}^*, S^*, T^*)$ would have smaller E{N} than that of Bofinger's procedure, at least for $s = k - 1$ when $p_{[k]} + p_{[k-1]} < 1$. (See our Conjecture 7.1.) However, we have made no computations to verify this assertion.

8.2 Procedures having different P{CS} from (R^*, S^*, T^*)

a) Sobel and Weiss [1972a] proposed open sequential procedures for $k \geq 2$ employing VT and several PW sampling rules all of which employed a stopping rule based on inverse sampling. All of the Sobel-Weiss (S-W) procedures stopped sampling as soon as any population produced r successes. The constant r was chosen to guarantee the $\{\Delta^*, P^*\}$ probability requirement, and was shown to have the same value for all of their procedures. (Berry and Young [1977] proved that these procedures using the same r achieved the same P{CS} uniformly in $\underset{\sim}{p}$.) However, if r of the S-W procedures and n of the Sobel-Huyett (S-H) procedure were chosen to guarantee the same $\{\Delta^*, P^*\}$ requirement, then the P{CS} achieved by these procedures is not the same uniformly in $\underset{\sim}{p}$. If (R^*, S^*, T^*) is used with the S-H n, then from Theorem 5.1 we have that the P{CS} achieved by (R^*, S^*, T^*) and (R_{SS}, T_{SS}) is the same uniformly in $\underset{\sim}{p}$, but this P{CS} differs from that of the S-W procedures which use the corresponding r. This fact makes it difficult to compare the E{N} - values for (R^*, S^*, T^*) with those of the S-W procedures.

If $p_{[1]} \to 1$ then the S-W procedures which use one-at-a-time sampling have smaller $E\{N\}$ than does the corresponding (R^*,S^*,T^*). However, if $p_{[k]} \to 0$ then $E\{N\} \to \infty$ for the S-W procedures while $N \leq kn - 1$ for (R^*,S^*,T^*).

b) Berry and Sobel [1973] proposed a closed sequential procedure for $k = 2$ employing a PW sampling rule. Sampling stops when either population produces r successes or both produce at least c failures. The constants (r,c) were chosen to guarantee the $\{\Delta^*,P^*\}$ requirement; Berry and Sobel (B-S) recommend the choice $r = c$ as optimal. As with the S-W procedure, the B-S procedure has a $P\{CS\}$ - function which differs from that of (R_{SS},T_{SS}) if $r = c$ and n are chosen to guarantee the same $\{\Delta^*,P^*\}$ requirement. A fortiori the B-S procedure has a different $P\{CS\}$ - function than does (R^*,S^*,T^*) which uses the n of (R_{SS},T_{SS}).

The B-S procedure has very desirable $E\{N\}$ behavior relative to that of (R^*,S^*,T^*), but less desirable $P\{CS\}$ behavior. When we determined n and r corresponding to the nine (Δ^*,P^*) combinations for $\Delta^* = 0.1, 0.2, 0.3$ and $P^* = 0.90, 0.95, 0.99$ and set $p_{[2]} - p_{[1]} = \Delta^*$ with $p_{[2]}$ varying we found that $E\{N|(B\text{-}S)\} < E\{N|(R^*,S^*,T^*)\}$ except in a small neighborhood of $(p_{[1]},p_{[2]}) = (1/2 - \Delta^*/2, 1/2 + \Delta^*/2)$ where $P\{CS|(R^*,S^*,T^*)\}$ achieves its minimum in the preference zone. (Note: $P\{CS|(B\text{-}S)\}$ achieves its minimum in the preference zone at $(p_{[1]},p_{[2]}) = (1/3 - \Delta^*/2, 1/3 + \Delta^*/2)$ and at $(2/3 - \Delta^*/2, 2/3 + \Delta^*/2)$; this result is an asymptotic $(P^* \to 1)$ one.) However, $P\{CS|(B\text{-}S)\} < P\{CS|(R^*,S^*,T^*)\}$ except in approximately this same small neighborhood. Thus it appears that the decrease in $E\{N\}$ for B-S is purchased at the cost of a decrease in $P\{CS\}$. (Small changes in $P\{CS\}$ result in large changes in $E\{N\}$ when the $P\{CS\}$ is close to unity.) We also found that $\max\{N|(B\text{-}S)\} > \max\{N|(R^*,S^*,T^*)\}$ for most $\{\Delta^*,P^*\}$ of interest.

c) Schriever [1978/79] generalized the B-S procedure to $k > 2$. His procedure which employs a PW sampling rule stops sampling when any population produces r successes or every population produces at least c failures. The constants (r,c) are chosen to guarantee the $\{\Delta^*,P^*\}$ requirement; Schriever too recommended the choice $r = c$ as optimal. At this time we do not have sufficient exact calculations to compare the performance of Schriever's procedure with that of (R^*,S^*,T^*) for $k > 2$.

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10. References

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Appendix: Proof of Theorem 5.1.

We shall prove Theorem 5.1 for the case $s = 1$; the proof for $s > 1$ proceeds in a similar manner. In this Appendix we denote $p_{[i]}$ by p_i, i.e., $p_1 \leq p_2 \leq \ldots \leq p_k$ are the ordered p's. For $0 \leq m \leq kn$ define

$$\Omega_m^n = \{x_m = (y_{1,m},n_{1,m};\ldots;y_{k,m},n_{k,m}): 0 \leq y_{i,m} \leq n_{i,m}, \sum_{i=1}^{k} n_{i,m} = m\}$$

where $y_{i,m}$ and $n_{i,m}$ are the number of successes and number of observations, respectively, from π_i through stage m $(1 \leq i \leq k)$. (Here π_i has probability of success p_i.) To emphasize the dependence of $y_{i,m}$ and $n_{i,m}$ on x_m we shall use the notation:

$$y_i(x_m) = y_{i,m}, \quad n_i(x_m) = n_{i,m} \quad (1 \leq i \leq k).$$

For $0 \leq m \leq kn$ define the set of stopping states at stage m by

$$S_m^n = \{x_m \in \Omega_m^n: \exists\ i \text{ s.t. } y_i(x_m) \geq y_j(x_m) + n - n_j(x_m)\ \forall\ j \neq i\}$$

and the set of continuation states at stage m by $C_m^n = \Omega_m^n \setminus S_m^n$.

Let $$S^n = \bigcup_{m=0}^{kn} S_m^n \quad \text{and} \quad C^n = \bigcup_{m=0}^{kn} C_m^n.$$

Define the probability of reaching state x_m along any feasible path using sampling rule R by

$$P_R(x_m) = a_R(x_m)P(x_m)$$

where

$$P(x_m) = \prod_{i=1}^{k} p_i^{y_i(x_m)} (1-p_i)^{n_i(x_m)-y_i(x_m)}$$

and $a_R(x_m)$ is the randomization coefficient employed by R. If it is not possible to reach state x_m using R, then it is clear that $P_R(x_m) = 0$. In this situation define $a_R(x_m) = 0$. Note that $a_R(x_m)$ depends only on the rule R and in particular is independent of the p_i's; $a_R(x_m)$ can and usually does depend on the data.

Example A.1: Suppose that R is PW for $k = 2$ with the following modification: if at any stage the number of successes and failures is the same for both populations, then take the next observation from one of them at random.

Let $n = 4$, $x_5 = (2,3; 1,2)$. It is possible to reach state x_5 along different sampling paths; two such paths are: $s_1 = S_2^1 F_2^2 S_1^3 S_1^4 F_1^5$ and $s_2 = S_1^1 F_1^2 S_2^3 F_2^4 S_1^5$.

Now,

$$P\{\text{reaching state } x_5 \text{ along } s_1\} = \frac{1}{2} p_2(1-p_2)p_1^2(1-p_1) = \frac{1}{2} P(x_5) \quad (A.1)$$

and

$$P\{\text{reaching state } x_5 \text{ along } s_2\} = \frac{1}{2} p_1(1-p_1)p_2(1-p_2)\frac{1}{2}p_1 = \frac{1}{4} P(x_5). \quad (A.2)$$

Thus we see that for the particular paths of our example the coefficients in (A.1) and (A.2), above, depend on the sampling rule R and the path followed. For any particular R, $a_R(x_m)$ is the sum of all such coefficients. To illustrate this latter point consider the following example.

Example A.2: Let $R = R_{SS}$. Here

$$P_R(x_{kn}) = a_R(x_{kn})P(x_{kn})$$

where

$$P(x_{kn}) = \prod_{i=1}^{k} p_i^{y_i(x_{kn})} (1-p_i)^{n-y_i(x_{kn})}$$

and

$$a_R(x_{kn}) = \prod_{i=1}^{k} \binom{n}{y_i(x_{kn})}.$$

Let $K = \{1,2,\ldots,k\}$ and $A \subseteq K$, $A \neq \emptyset$. Let W_A denote the set of all stopping states, x_{kn}, which would lead to selection at random from the populations in A, using (R_{SS},T_{SS}), i.e.,

$$W_A = \{x_{kn} \in \Omega_{kn}^n: y_i(x_{kn}) = \max_{1\leq j\leq k} y_j(x_{kn}) \ \forall\, i \in A$$

$$y_i(x_{kn}) < \max_{1\leq j\leq k} y_j(x_{kn}) \ \forall\, i \notin A\}.$$

Next let V_A denote the set of all stopping states, x_m, which for some m lead to selection at random from the populations in A, using (R,S^*,T^*), i.e.,

$$V_A = \{x_m \in S_m^n \ (1 \leq m \leq kn): y_i(x_m) \geq y_j(x_m) + n - n_j(x_m) \ \forall\, j \neq i \text{ iff } i \in A\}$$

Let $x_m \in \Omega_m^n$ $(0 \leq m \leq kn)$ and $x_{kn} \in \Omega_{kn}^n$. We will say that $x_m \subset x_{kn}$ if it is possible to augment x_m by the remaining observations which have not yet been taken to obtain x_{kn}, i.e., if

$$0 \leq y_j(x_{kn}) - y_j(x_m) \leq n - n_j(x_m) \quad (1 \leq j \leq k).$$

Example A.3: Let $k = 2$, $n = 2$ and suppose that $x_2 = (0,1; 1,1) \in \Omega_2^2$, and $x_4 = (1,2; 1,2) \in \Omega_4^2$. Then $x_2 \subset x_4$ since

$$0 \leq y_j(x_4) - y_j(x_2) \leq 2 - n_j(x_2) \quad (j = 1,2).$$

Let $x_m \subset x_{kn}$. Define the 'difference' between the original state and the augmented state by

$$d(x_m,x_{kn}) = \{y_1(d(x_m,x_{kn})),n_1(d(x_m,x_{kn}));\ldots;$$

$$\ldots;y_k(d(x_m,x_{kn})),n_k(d(x_m,x_{kn}))\}$$

where

$$\begin{cases} y_j(d(x_m,x_{kn})) = y_j(x_{kn}) - y_j(x_m) \\ n_j(d(x_m,x_{kn})) = n - n_j(x_m) \end{cases} \quad (1 \leq j \leq k).$$

Thus,

$$P(d(x_m,x_{kn})) = \prod_{i=1}^{k} p_i^{y_i(x_{kn})-y_i(x_m)} (1-p_i)^{n-n_j(x_m)-y_i(x_{kn})+y_i(x_m)}$$

and

$$P(x_m)P(d(x_m,x_{kn})) = \prod_{i=1}^{k} p_i^{y_i(x_{kn})} (1-p_i)^{n-y_i(x_{kn})}$$

$$= P(x_{kn}). \qquad (A.3)$$

Since p_k is the largest of the p's and $k \in A$ implies correct selection with probability $1/|A|$, we can write

$$P\{CS|(R_{SS},T_{SS})\} = \sum_{A:\ A\subseteq K,\ k\in A} \sum_{x_{kn}\in W_A} \frac{1}{|A|} P_{R_{SS}}(x_{kn}) \qquad (A.4)$$

and

$$P\{CS|(R,S^*,T^*)\} = \sum_{A:\ A\subseteq K,\ k\in A} \sum_{x_m\in V_A} \frac{1}{|A|} P_R(x_m). \qquad (A.5)$$

For $x_m \in S_m^n$ let $E^n(x_m)$ denote the set of all possible augmentations x_{kn} of x_m, i.e.,

$$E^n(x_m) = \{x_{kn} \in \Omega_{kn}^n :\ x_m \subset x_{kn}\}.$$

To prove Theorem 5.1 we shall use the following lemmas.

Lemma A.1. Let $x_m \in S_m^n$. Then,

$$\sum_{x_{kn}\in E^n(x_m)} \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} P(d(x_m,x_{kn})) = 1.$$

Proof: When we consider all possible augmentations of x_m we have k independent binomial distributions with parameters $(n-n_i(x_m),p_i)$ $(1 \le i \le k)$. Hence, l.h.s.

$$= \sum_{y_1=0}^{n-n_1(x_m)} \cdots \sum_{y_k=0}^{n-n_k(x_m)} \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i} p_i^{y_i}(1-p_i)^{n-n_i(x_m)-y_i}$$

$$= \prod_{i=1}^{k} \sum_{y_i=0}^{n-n_i(x_m)} \binom{n-n_i(x_m)}{y_i} p_i^{y_i}(1-p_i)^{n-n_i(x_m)-y_i}$$

$$= 1.$$

□

Lemma A.2. Let $x_m \in V_A$ and suppose that $x_m \subset x_{kn}$. Then $x_{kn} \in \bigcup_{B \supseteq A} W_B$. In other words, if x_m is a stopping state using R which leads to selection among the populations in A, then if x_m is augmented to obtain x_{kn}, we will at most randomize among the elements of a superset of A.

Proof: $x_m \in V_A$

$$\Rightarrow y_i(x_m) > y_j(x_m) + (n - n_j(x_m)) \quad \forall\, j \neq i,\ \forall\, i \in A.$$

Hence, $x_m \subset x_{kn}$

$$\Rightarrow y_i(x_{kn}) \geq y_i(x_m)$$

$$\geq y_j(x_m) + (n - n_j(x_m)) \quad \forall\, j \neq i$$

$$> y_j(x_m) + (y_j(x_{kn}) - y_j(x_m))$$

$$= y_j(x_{kn}),$$

$$\text{i.e.,} \quad y_i(x_{kn}) \geq y_j(x_{kn}) \quad \forall\, j \neq i,\ \forall\, i \in A$$

and possibly for some $i \notin A$. Thus it follows that $y_i(x_{kn}) = \max_{1 \leq j \leq k} y_j(x_{kn})$, at least for all $i \in A$. Hence, $x_{kn} \in W_B$ for some $B \supseteq A$. Therefore,

$$x_m \subset x_{kn} \Rightarrow x_{kn} \in \bigcup_{B \supseteq A} W_B. \qquad \square$$

Note: If $x_{kn} \in W_A$ and if $x_m \in S^n$ is s.t. $x_m \subset x_{kn}$ for some m, then $x_m \in \bigcup_{B \subseteq A} V_B$.

Lemma A.3. Let $x_{kn} \in \Omega^n_{kn}$ and suppose R is any sampling rule which takes no more than n observations per population. Then,

$$\sum_{x_m \subset x_{kn},\ x_m \in S^n} a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} = \prod_{i=1}^{k} \binom{n}{y_i(x_{kn})}.$$

Proof: Consider an arbitrary state x_{kn} of outcomes (successes and failures) to which (R_{SS},T_{SS}) is applied. Also consider the possible stopping states $x_m \subset x_{kn}$. If (R,S^*,T^*) is used instead of (R_{SS},T_{SS}) for the same x_{kn}, we will stop at some stopping state $x_m \subset x_{kn}$. Clearly,

$$P\{\text{termination using } (R,S^*,T^*) | x_{kn} \text{ using } R_{SS}\} = 1$$

$$\Rightarrow \sum_{x_m \subset x_{kn}} \frac{P\{\text{terminate at state } x_m;\ \text{complete } x_m \text{ to } x_{kn}\}}{P_{R_{SS}}(x_{kn})} = 1,$$

$$\text{i.e., } \sum_{x_m \subset x_{kn}} \frac{P\{\text{terminate at state } x_m\} P\{\text{complete } x_m \text{ to } x_{kn} | x_m\}}{P_{R_{SS}}(x_{kn})} = 1,$$

$$\text{i.e., } \sum_{x_m \subset x_{kn}} \frac{P_R(x_m)\left[\prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} P(d(x_m,x_{kn}))\right]}{P_{R_{SS}}(x_{kn})} = 1,$$

$$\text{i.e., } \sum_{x_m \subset x_{kn}} \frac{a_R(x_m)P(x_m)\left[\prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} P(d(x_m,x_{kn}))\right]}{\prod_{i=1}^{k} \binom{n}{y_i(x_{kn})} P(x_{kn})} = 1,$$

$$\text{i.e., } \sum_{x_m \subset x_{kn}} \frac{a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} P(x_m)P(d(x_m,x_{kn}))}{\prod_{i=1}^{k} \binom{n}{y_i(x_{kn})} P(x_{kn})} = 1.$$

Using equation (A.3) we obtain

$$\sum_{x_m \subset x_{kn}} \frac{a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} P(x_{kn})}{\prod_{i=1}^{k} \binom{n}{y_i(x_{kn})} P(x_{kn})} = 1$$

Thus $$\sum_{x_m \subset x_{kn}} a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} = \prod_{i=1}^{k} \binom{n}{y_i(x_{kn})}.$$

□

Lemma A.4. Let $x_{kn} \in W_B$, $k \in B$. Then,

$$\sum_{\substack{A: A \subseteq B, \\ k \in A}} \frac{1}{|A|} \sum_{\substack{x_m : x_m \subset x_{kn}, \\ x_m \in V_A}} a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} = \frac{1}{|B|} \prod_{i=1}^{k} \binom{n}{y_i(x_{kn})}. \tag{A.6}$$

Proof: If $|B| = 1$, then the Lemma is true by Lemma A.3, since $A \subseteq B$, $k \in A \Rightarrow A = B$ and $x_m \subset x_{kn} \Rightarrow x_m \in V_A = V_B$.

Suppose $|B| > 1$.

Let $r \in B$, $r \neq k$. Now, $r,k \in B$ and $x_{kn} \in W_B$

$$\Rightarrow y_r(x_{kn}) = y_k(x_{kn}) = \max_{1 \leq j \leq k} y_j(x_{kn}).$$

Since $y_r(x_{kn}) = y_k(x_{kn})$, and recalling that the pairing of the $p_{[i]}$ with the Π_j $(1 \leq i,j \leq k)$ is completely unknown (and that the populations are tagged in such a way that their ordering is unique), it can be seen that interchanging the labels r and k would lead to the same stopping states contained in x_{kn} with the same randomization coefficients. Thus

$$\sum_{\substack{A: A \subseteq B, \\ k \in A}} \frac{1}{|A|} \sum_{\substack{x_m \subset x_{kn}, \\ x_m \in V_A}} a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} \tag{A.7}$$

$$= \sum_{\substack{A: A \subseteq B, \\ r \in A}} \frac{1}{|A|} \sum_{\substack{x_m \subset x_{kn}, \\ x_m \in V_A}} a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)}.$$

Hence, the l.h.s. of (A.6)

$$= \frac{1}{|B|} \sum_{r \in B} \sum_{\substack{A: A \subseteq B, \\ k \in A}} \frac{1}{|A|} \sum_{\substack{x_m \subset x_{kn}, \\ x_m \in V_A}} a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)}$$

which using (A.7)

$$= \frac{1}{|B|} \sum_{r \in B} \sum_{\substack{A: A \subseteq B, \\ r \in A}} \frac{1}{|A|} \sum_{\substack{x_m \subset x_{kn}, \\ x_m \in V_A}} a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)}.$$

The above equals

$$\frac{1}{|B|} \sum_{A: A \subseteq B} \sum_{r \in A} \frac{1}{|A|} \sum_{\substack{x_m \subset x_{kn}, \\ x_m \in V_A}} a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)}$$

$$= \frac{1}{|B|} \sum_{A: A \subseteq B} \sum_{\substack{x_m \subset x_{kn}, \\ x_m \in V_A}} a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)}$$

$$= \frac{1}{|B|} \sum_{x_m \subset x_{kn}} a_R(x_m) \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)}$$

by the note following Lemma A.2,

$$= \frac{1}{|B|} \prod_{i=1}^{k} \binom{n}{y_i(x_{kn})}$$

by Lemma A.3. □

We now proceed with our proof of Theorem 5.1.

Proof: From equation (A.5) we have

$$P\{CS|(R,S^*,T^*)\} = \sum_{\substack{A:\ A\subseteq K,\\ k\in A}} \frac{1}{|A|} \sum_{x_m\in V_A} P_R(x_m)$$

$$= \sum_{\substack{A:\ A\subseteq K,\\ k\in A}} \frac{1}{|A|} \sum_{x_m\in V_A} a_R(x_m)P(x_m).$$

By Lemma A.1 we can write

$$P\{CS|(R,S^*,T^*)\}$$

$$= \sum_{\substack{A:\ A\subseteq K,\\ k\in A}} \frac{1}{|A|} \sum_{x_m\in V_A} a_R(x_m)P(x_m) \sum_{x_{kn}\in E^n(x_m)}$$

$$\cdot \left\{ \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} P(d(x_m,x_{kn})) \right\}$$

$$\sum_{\substack{A:\ A\subseteq K,\\ k\in A}} \frac{1}{|A|} \sum_{x_m\in V_A} a_R(x_m) \sum_{B:\ B\subseteq K} \sum_{\substack{x_{kn}\in W_B,\\ x_{kn}\neq x_m}}$$

$$\cdot \left\{ \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} P(x_m)P(d(x_m,x_{kn})) \right\}.$$

As a consequence of Lemma A.2 the l.h.s. equals

$$\sum_{\substack{A:\ A\subseteq K,\\ k\in A}} \frac{1}{|A|} \sum_{x_m\in V_A} a_R(x_m) \sum_{B:\ B\subseteq K} \sum_{\substack{x_{kn}\in W_B\\ x_{kn}\neq x_m}}$$

$$\cdot \left\{ \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} P(x_m)P(d(x_m,x_{kn})) \right\}$$

which using equation (A.3) can be written as

$$= \sum_{\substack{A\subseteq K,\\ k\in A}} \; \frac{1}{|A|} \sum_{x_m \in V_A} a_R(x_m) \sum_{B:\ A\subseteq B} \; \sum_{\substack{x_{kn}\in W_B,\\ x_{kn}\neq x_m}} \cdot \left\{ \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} P(x_{kn}) \right\}$$

$$= \sum_{\substack{B\subseteq K\\ k\in B}} \; \sum_{x_{kn}\in W_B} \; \sum_{A:\ A\subseteq B,\ k\in A} \; \sum_{\substack{x_m:\ x_m\neq x_{kn},\\ x_m\in V_A}} \frac{1}{|A|}\, a_R(x_m) \cdot \left\{ \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)} P(x_{kn}) \right\}$$

$$= \sum_{\substack{B\subseteq K\\ k\in B}} \; \sum_{x_{kn}\in W_B} P(x_{kn}) \sum_{\substack{A:\ A\subseteq B\\ k\in A}} \frac{1}{|A|} \sum_{\substack{x_m:\ x_m\neq x_{kn},\\ x_m\in V_A}} a_R(x_m) \cdot \prod_{i=1}^{k} \binom{n-n_i(x_m)}{y_i(x_{kn})-y_i(x_m)}$$

Finally, using Lemma 3.4, this reduces to

$$= \sum_{\substack{B\subseteq K,\\ k\in B}} \; \sum_{x_{kn}\in W_B} P(x_{kn}) \frac{1}{|B|} \prod_{i=1}^{k} \binom{n}{y_i(x_{kn})} = \sum_{\substack{B\subseteq K,\\ k\in B}} \frac{1}{|B|} \sum_{x_{kn}\in W_B} P_{R_{SS}}(x_{kn})$$

$$= P(C_k|(R_{SS},T_{SS})). \quad \square$$

 Block No. 20 Continued

single-stage procedure of Sobel and Huyett [1957] which takes exactly n observations from each of the k populations. We propose a one-at-a-time adaptive sampling rule (R^*) which when used in conjunction with a particular stopping rule (S^*) and terminal decision rule (T^*) achieves the same probability of a correct selection as does the single-stage procedure uniformly in $\underset{\sim}{p} = (p_1,\ldots,p_k)$. Letting N denote the random total number of observations to terminate sampling using the procedure (R^*,S^*,T^*) we show that $n \leq N \leq kn-1$; for $p_{[k]} \to 0$ we have $P\{N = kn-1 \mid \underset{\sim}{p}\} \to 1$ while for $p_{[1]} \to 1$ we have $P\{N = n \mid \underset{\sim}{p}\} \to 1$. For $k = 2$ the sampling rule R^* (the conjugate sampling rule $\overline{R}^*$) which is stationary is optimal in the sense that it minimizes $E\{N|(p_1,p_2)\}$ uniformly in (p_1,p_2) for $p_1 + p_2 > 1$ $(p_1 + p_2 < 1)$ among all sampling rules which use (S^*,T^*) and which take no more than n observations from either population; R^* has additional optimal properties for $k = 2$. The procedure (R^*,S^*,T^*) is generalized for $k > 2$ to accommodate such goals as "Selecting the s $(1 \leq s \leq k-1)$ "best" Bernoulli populations with regard to order," and is shown to have desirable properties for these goals as well. Some conjectures are made concerning the optimality of (R^*,S^*,T^*) for $k > 2$. The performance of (R^*,S^*,T^*) is compared for $k \geq 2$ with that of other sequential selection procedures that have been proposed in the literature. An extensive bibliography is included.

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The goal of selecting that one of $k \geq 2$ Bernoulli populations which has the largest single-trial "success" probability $p_{[k]} = \max(p_1, \ldots, p_k)$ is treated. Consideration is restricted to procedures which take no more than n observations from any one of the k populations. One such procedure is the

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